Abstract—A new toroidal field (TF) coil whose stress is much reduced is proposed for tokamaks by use of the virial theorem. It is a hybrid coil of a conventional TF coil and an ohmic heating (OH) coil helically wound on a torus. The winding is modulated in such a way that the poloidal field exists only outside of the torus, because the poloidal field in the torus prevents the breakdown of plasma and causes the torsional force. According to the virial theorem, the best TF coil to produce the strongest magnetic field under the weakest averaged stress requires equal averaged principal stresses of all directions. Therefore, the pitch number of the helical coil is determined to satisfy the uniform stress condition. The enhancement factor of the magnetic field in our TF coil increases as the aspect ratio $A$ decreases where the distribution of the stress by the toroidal effect is important. In the case of $A = 2$, our optimal coil increases the magnetic field to 1.4 times larger than the traditional TF coil. From a viewpoint of stress, it reduces the maximum stress to the half in the traditional one.

Keywords—fusion, tokamak, coil, virial theorem, magnetic field, stress, tensor, energy

I. INTRODUCTION

The virial theorem shows the relation between long time average or constant value of the kinetic energy and the energy of the field. Famous examples are the relation of the gravity potential and the kinetic energy in the astrophysics, and the relation between the kinetic energy and potential energy of the charged particles in the electromagnetics. In the field of the nuclear fusion, the relations between the volume integration and surface integration of the physical quantity in plasma are called the virial theorem[1]. This theorem shows that there is no equilibrium with only the pressure and that the tension is necessary to hold the balance of force. Furthermore, the virial theorem for the general electromagnetic structure is obtained when the pressure of the plasma in the balance equation is replaced to the stress.

We have designed the tokamak with force balanced coil (FBC)[2] which is a helical type hybrid coil combined the toroidal field (TF) coil and center solenoidal coil. The combination removes the net electromagnetic force in the major radius direction[3], [4] by canceling the centering force by the TF coil and the hoop force by the solenoidal coil. Furthermore, we showed the configuration without poloidal magnetic field inside torus by giving poloidal dependence to the pitch angle of the helical winding[5]. Moreover, we made a tokamak using FBC and proved the plasma production and confinement in it[6]-[9], experimentally.

In this work, we extend and generalize our studies with the virial theorem, and show the way to minimize the stress working in the coil. Furthermore, we design the coil with minimum stress under the condition of the fixed toroidal field. It means that this design enable us to make a coil with less amount of the structure to support the coil. In the next section, we explain the virial theorem for the structure with the electromagnetic force. In section III, we apply the virial theorem to the thin toroidal shell, and show the optimal coil for stress. The toroidal effect is investigated in section IV, and the uniaxial model is applied in section V to check the results of the shell model. Finally, this work is summarized in section VI.

II. VIRIAL THEOREM OF ELECTROMAGNETIC STRUCTURE

When electromagnetic force and stress are balanced in the object $\Omega$, the following equilibrium equations hold,

$$j \times B + \nabla \cdot S = 0,$$
$$\nabla \times B = \mu_0 j,$$
$$\nabla \cdot B = 0,$$

where $j$ is the current density, $B$ is the magnetic flux density, $S$ is the stress tensor. From (1)-(3) with Gauss’s integral theorem, the next equation is ob-
\[ \int \sum_i \sigma_i dV = \int \frac{B^2}{2\mu_0} dV = U_M, \quad (4) \]

where \( \sigma_i (i = 1, 2, 3) \) are the principal stresses. We call this relation as the virial theorem[10]. Because the right hand side (the magnetic energy \( U_M \)) is positive, the positive stress (the tensile stress) is necessary to make the magnetic field. Furthermore, to minimize the positive stress (the tensile stress) is necessary to decrease the maximum stress under the condition of a fixed magnetic energy and volume.

Hence it is clear that the uniform stress distribution is required to decrease the maximum stress under the condition of a fixed magnetic energy and volume.

When we define the maximum stress in the object \( \Omega \), the next equation is obtained[10],

\[ \begin{align*}
\sigma_{\text{max}} & = \frac{\rho_m U_M}{3 \sigma_{\text{max}}} , \quad (8) \\
M & \geq \frac{\rho_m U_M}{3 \sigma_{\text{max}}} ,
\end{align*} \]

where \( M, \rho_m \) are the mass and the density of the object \( \Omega \), respectively. The equality of (8) holds only when three principal stresses have the same value and uniform in the object \( \Omega \). In the other case, stress distribution is necessary to evaluate the minimum of mass \( M \) precisely.

### III. Application to Thin Toroidal Shell

In this section, we analyze the shape of TF coil, and develop the theory to include FBC, which balances the electromagnetic force of major radius direction.

First of all, we consider the axisymmetric toroidal shell with circular cross section whose thickness \( \Delta \rho \) is much small than the major radius \( R \) and the minor radius \( a \). The current distribution on the torus is determined so that a magnetic surface coincides with the torus. Moreover, we assume that the material of torus is uniform and \( A = R/a \) is much larger than unity. Under the assumptions, all physical quantities are uniform on the torus. In this work, we choose semi-toroidal coordinate system \((\rho, \phi, \theta)\). Under the above assumptions, the principal stresses are \( \sigma_\rho, \sigma_\phi, \sigma_\theta \), and next relation holds,

\[ \sigma_\rho \ll \sigma_\phi, \sigma_\theta . \quad (9) \]

Therefore, we set \( \sigma_\rho = 0 \) in this paper. In the case that the aspect ratio \( A = R/a \) is large enough, the magnetic energy \( U_M \) of the toroidal coil is given by

\[ \begin{align*}
U_M & = U_{\text{TF}} + U_{\text{PF}}, \\
U_{\text{TF}} & = \frac{\mu_0 A^2}{4R} I_\phi^2 , \\
U_{\text{PF}} & = \frac{\mu_0 R}{2} \left( \log \frac{8R}{a} - 2 \right) I_\phi^2 , \quad (12)
\end{align*} \]

where \( U_{\text{TF}} \) and \( U_{\text{PF}} \) are the energy of the toroidal and poloidal field inside the torus, \( I_\phi \) and \( I_\theta \) are the toroidal and poloidal current, respectively. The net electromagnetic force \( F_R, F_\theta \) in major and minor radius directions are obtained from the partial differentiations of magnetic energy \( U_M \) with respect to \( R \) and \( a \).

\[ \begin{align*}
F_R & = \frac{\partial U_M}{\partial R} \bigg|_{I=\text{const.}} \\
& = \frac{\mu_0}{2} \left( (\log 8A - 1) I_\phi^2 - \frac{1}{2A^2} I_\theta^2 \right) , \quad (13) \\
F_\theta & = \frac{\partial U_M}{\partial a} \bigg|_{I=\text{const.}} \\
& = \frac{\mu_0}{2} \left( \frac{1}{A} I_\theta^2 - AI_\phi^2 \right) . \quad (14)
\end{align*} \]

According to the principle of virtual work, we obtain the normalized stress defined in (5) as follows:

\[ \begin{align*}
\langle \tilde{\sigma}_\phi \rangle & = \frac{A^2 \log 8A - A^2 - \frac{N_\theta^2}{2}}{\Delta \rho^2 + A^2 \log 8A - 2A^2} , \quad (15) \\
\langle \tilde{\sigma}_\theta \rangle & = \frac{N_\theta^2 - A^2}{\Delta \rho^2 + A^2 \log 8A - 2A^2} , \quad (16)
\end{align*} \]

where

\[ N = \frac{I_\theta}{I_\phi} \quad (17) \]

is the ratio of the toroidal current and the poloidal current, which means the pitch number of a coil. In the limit of \( N \to \infty \), they become \( \tilde{\sigma}_\phi = -1, \tilde{\sigma}_\theta = 2 \) which are the values of the conventional TF coil. When we get the sum of the normalized stresses with (9), (15) and (16),

\[ \langle \tilde{\sigma}_\phi \rangle + \langle \tilde{\sigma}_\theta \rangle = 1 , \]
the virial theorem (7) is satisfied. From (15), (16), the average stress in the toroidal direction or poloidal direction vanishes in the case of

\[ N^2 = 2A^2(\log 8A - 1) \quad (\sigma_\phi = 0), \]  
\[ N = A \quad (\sigma_\theta = 0). \]

respectively. In particular, the coil with pitch number of (18) is called the force-balanced coil (FBC)[3]. Because we assume the uniformity of the structure in this work, the configuration with the minimum stress is established at

\[ \langle \bar{\sigma}_\phi \rangle = \langle \bar{\sigma}_\theta \rangle = \frac{1}{2}, \]

while the configuration takes the following pitch number,

\[ N^2 = \frac{2}{3}A^2 \log 8A. \]

Figure 1 represents the relation of the aspect ratio and the pitch number, and shows that the configuration with minimum stress exists between two types of FBCs.

Next we define the stress \( \dot{\sigma} \) normalized by the toroidal magnetic energy \( U_{TF} \) instead of the total magnetic energy \( U_M \) in order to compare the strength of toroidal magnetic field,

\[ \dot{\sigma} \equiv \frac{V_\Omega}{U_{TF}} \sigma. \]

Since the magnetic energy is proportional to the square of the strength of magnetic field, \( 1/\sqrt{\sigma} \) is proportional to the strength of magnetic field. Thus the parameter,

\[ \frac{B_\phi}{B_{TFC}} = \sqrt{\frac{\sigma}{\dot{\sigma}}}, \]

is the enhanced factor of toroidal field with the same shape and stress. The toroidal fields evaluated by (23) in the three kinds of coils with the same shape and maximum stress are compared in Fig. 2. Our virial coil can produce about 1.5 times strong magnetic field compared with that of the conventional TF coil in the range of the small aspect ratio.

IV. Toroidal Shell with Circular Cross Section

Although we have investigated the case that the aspect ratio \( A \) is sufficiently large in the preceding section, we are not able to ignore the toroidal effect because \( A \) of an actual tokamak device is about 3. The magnetic energies of (11) and (10), which are used in determining the pitch number \( N \), also assume that \( A \) is sufficiently large. Furthermore, as is shown in Fig. 2, it is in the case of the small aspect ratio that our helical coil can make sufficiently stronger magnetic field than TF coil. Therefore, we evaluate the poloidal distribution of stress to investigate the influence to the stress by the toroidal effect in this section.

Supposing that the pressure inside torus is \( p(r) \) larger than that of the outside, the force balance equations of the infinitesimal volume \( \Delta r dr d\phi d\Delta \rho \) for the directions of \( \rho, \theta \) are obtained. Using the tensions for the unit length \( T_\theta, T_\phi \), we obtain the following equations,

\[ rT_\theta + (r - R)T_\phi = arp(r), \]
\[
\frac{d}{dr}(rT_\theta) = T_\phi. \tag{25}
\]

To solve this differential equation system, we define the function \( u \),
\[
u(r) = a \int_R^r r'p(r')dr', \tag{26}\]
and get the solution as follows:
\[
T_\theta = \frac{u}{(r-R)r}, \tag{27}
\]
\[
T_\phi = \frac{arp}{(r-R)} - \frac{u}{(r-R)^2}. \tag{28}\]

These equations enable us to calculate the stress distribution on a toroidal shell with a circular cross section whose current layer coincides with the magnetic surface. When we obtain the toroidal current distribution numerically, the pressure outside the torus is obtained from the toroidal surface current density \( j_\phi \). Hence the magnetic pressure acting on the torus is given by
\[
p = \frac{\mu_0 I^2_\theta}{8\pi^2 r^2} - \frac{\mu_0}{2} J^2_\phi. \tag{29}\]

Next, we calculate the stress distribution of the helical coil with \( A = 2 \) using (27)-(29). The results for some kinds of coils are shown in Fig. 3. To compare them in the condition of a fixed toroidal field, they are normalized by the toroidal magnetic energy \( U_{TF} \). The poloidal angle \( \theta \) is defined so as to have 0 at the outside of the torus and have \( \pi \) at the inside of the torus. Although the virial coil (solid line) is expected to decrease the maximum stress in comparison with TF coil (dashed line), it does not decrease as is expected in Fig. 2. Therefore, we decide the pitch number \( N \) so that the principal stresses have the same value at the point of \( \theta = \pi \) where the stress has the maximum value. The stress distribution of the optimal coil is represented by dot-dashed line in Fig.3. It shows that the strength of the magnetic field is increased to about 1.4 times larger and the maximum stress is reduced to about half in comparison with those of TF coil. Therefore, in the case of low aspect ratio, we have to decide the optimal pitch number without (21) so that the stress at \( \theta = \pi \) is minimum.

**V. Uniaxial Stress Model**

Although we have assumed the axisymmetric and continuous nature, the actual helical coil does not satisfy the assumptions. Therefore, we investigate the distribution of stress in one-dimensional cable with flexural rigidity like a cable-in-conduit cable. For this purpose, we use the uniaxial model depicted in Fig.4. The equilibrium equations for the infinitesimal small region of the coil are written as
\[
\frac{dT}{ds} + \frac{F_u}{R_c} = 0, \tag{30}
\]
\[
\frac{dF_u}{ds} + \frac{T}{R_c} = f_u, \tag{31}
\]
\[
\frac{dF_v}{ds} = f_v, \tag{32}\]

where \( F \) is the sharing force, \( R_c \) is the radius of curvature, and \( f \) is the electromagnetic force. Solving this equation, we obtain Fig. 5 for \( A = 2.1 \). In Fig. 5, it is shown that our virial coil realizes nearly flattened distribution of stress.

**VI. Summary and Discussions**

Using the virial theorem, the relation between the toroidal magnetic field and stress is obtained. The tension is available to produce the magnetic field while the compression does not work to produce the mag-
Fig. 5. The distributions of tension for some kinds of coils for $R = 0.297\text{m}$, $a = 1.141\text{m}$, $B_\phi = 1.65\text{T}$ at $r = R$. TF coil and virial coil are represented by dashed and solid lines, respectively.

Next, we apply this result to the coil wound on the thin toroidal shell, and obtain the optimal pitch number of the coil which can produce the maximum toroidal field with the same maximum stress. The helical coil with the optimal pitch number can produce the toroidal field greater than the conventional TF coil in any aspect ratio. However, the poloidal distribution of the stress by the toroidal effect can not be ignored because the enhancement is large in the range of the small aspect ratio. When the aspect ratio is reduced to 2, the enhancement effect of the toroidal field increases and the poloidal averaged value of the enhancement factor to the conventional TF coil becomes about 1.5. In this case, our virial coil can produce the magnetic field of about 1.4 times larger than TF coil, even if the the distribution of the stress by the toroidal effect is not included. In other words, the maximum stress to produce the same toroidal field as that of TF coil becomes to about half.

In the actual coil, the assumptions of the two-dimensional continuous torus are not valid. However, using the uniaxial model, it is shown that our virial coil is available in the realistic system.

In this work, we have not yet included poloidal field (PF) coils and plasma, which make poloidal magnetic field and affect the distribution of stress. The control of PF coil current is also influenced by our virial coil because of the magnetic coupling. Therefore, we shall estimate their effects and extend this work.

**References**